Integration Technique (Algebraic Fraction)

1. Integration involving algebraic fraction (In the form $\frac{b}{\sqrt{a^2-x^2}}$, $\frac{b}{a^2+x^2}$, $\frac{f'(x)}{\sqrt{a^2-[f(x)]^2}}$, $\frac{f'(x)}{a^2+[f(x)]^2}$)

RECAP: Integration involving inverse Trigonometry

For integration involving $\frac{b}{\sqrt{a^2-x^2}}$ or $\frac{b}{a^2+x^2}$ or $\frac{f'(x)}{\sqrt{a^2-[f(x)]^2}}$ or $\frac{f'(x)}{a^2+[f(x)]^2}$, it is common that the use of inverse trigonometry would be involved:

	General	Function
a)	$\int \frac{b}{\sqrt{a^2-x^2}}dx = b\sin^{-1}\left(\frac{x}{a}\right) + C \;, \lvert x\rvert < a$	$\int \frac{f'(x)}{\sqrt{a^2-[f(x)]^2}}dx = \sin^{-1}\left(\frac{f(x)}{a}\right) + C$
b)	$\int \frac{b}{a^2+x^2}dx = \frac{b}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$	$\int \frac{f'(x)}{a^2+[f(x)]^2}dx = \frac{1}{a}\tan^{-1}\left(\frac{f(x)}{a}\right) + C$

When do we have to use this formulae?

- When the expression to be integrated has the form $\frac{f(x)}{\sqrt{ax^2+bx+c}}$ or $\frac{f(x)}{ax^2+bx+c}$ and ax^2+bx+c cannot be factorised
- For $\frac{f(x)}{ax^2+bx+c}$:
  - the relationship between $f(x)$ and ax^2+bx+c MUST NOT be a differential of each other
  - ax^2+bx+c cannot be factorised
- Completing the square is generally required for the above integration to work

Examples of expression requiring this formulae

$\int \frac{1}{\sqrt{9-x^2}}dx \to \int \frac{1}{\sqrt{(3)^2-x^2}}dx$ $a \to 3$ $f(x) \to x$	$\int \frac{1}{\sqrt{5-2x^2}}dx \to \frac{1}{\sqrt{2}}\int \frac{1}{\sqrt{\left(\sqrt{\frac{5}{2}}\right)^2 - x^2}}dx$ $a \to \sqrt{\frac{5}{2}}$ $f(x) \to x$
$\int \frac{1}{4+9x^2}dt \to \frac{1}{9}\int \frac{1}{\left(\frac{2}{3}\right)^2 + x^2}dx$ $a \to \frac{2}{3}$ $f(x) \to x$	$\int \frac{1}{x^2-2x+5}dx \to \int \frac{1}{(x-1)^2+(2)^2}dx$ $a \to 2$ $f(x) \to x-1$